v4 : 2016. 8. 10

2016 Fall

1. (20 pts.) Consider the following linear program:

$$\begin{aligned} \max \quad & 4x_1 + x_2 + 5x_3 + 3x_4 \\ \text{s.t.} \quad & x_1 - x_2 - x_3 + 3x_4 \le 1 \\ & 5x_1 + x_2 + 3x_3 + 8x_4 \le 55 \\ & -x_1 + 2x_2 + 3x_3 - 5x_4 \le 3 \end{aligned}$$

It is claimed that the solution $x^* = (0, 14, 0, 5)$ is an optimal solution to the problem. Give a proof of the claim. Do not use the simplex method to solve this problem.

2. (20 pts) Consider the linear program given in problem 1 again. After adding slack variables $x_5, x_6,$ and x_7 to each constraint respectively, simplex method is used to solve the problem. Then, the following optimal tableau is obtained:

$$\begin{array}{cccccccccc} -z & -x_1 & & -2x_3 & & -11x_5 & & -6x_7 & = & -29 \\ & 2x_1 & +x_2 & +4x_3 & & +5x_5 & & +3x_7 & = & 14 \\ & x_1 & & +x_3 & +x_4 & +2x_5 & & +x_7 & = & 5 \\ & -5x_1 & & -9x_3 & & -21x_5 & +x_6 & -11x_7 & = & 1 \end{array}$$

(a) (10pts) Now, the optimal set of basic variables is $\{x_2, x_4, x_6\}$. Find the inverse of the optimal basis matrix B. Explain how it can be obtained.

(b) (10pts) Suppose that the right hand side of the first constraint has been changed from 1 to $1 + \delta$. Find the range of δ in which the current basis remains optimal basis.

3. (30pts) Suppose that $Y_1, \ldots, Y_n$ denote a random sample from the Poisson distribution with mean λ, P(Y = y) = $e^{-\lambda}\lambda^{y}/y!$ for y = 0, 1, 2, ...

(a) Derive the MLE (maximum likelihood estimator) for λ.

(b) Show if the estimator of part (a) is consistent for λ.

(c) What is the MLE for P(Y = 0)?

4. (30 pts) Let X_n's be independent and nonnegative random variables. A random variable Y is exponentially distributed with rate λ and it is independent of X_n's.

(1) (10 pts) memoryless property: show that $P(Y > X_1 + X_2 | Y > X_1) = P(Y > X_2)$. Using this, conclude that $P(X_1 + \cdots + X_n < Y) = P(X_1 < Y)^n$.

(2) (10 pts) For a nonnegative and integer-valued random variable N, show

$$E[N] = \sum_{n=1}^{\infty} P(N \geq n).$$

(3) (10 pts) Define an integer value random variable $N_Y = \max\{n | X_1 + \cdots + X_n \leq Y\}$. If we interpret X_i's as inter-arrival times of customers, then N_Y is the number of arrivals within the random amount of time Y. Show that

$$E[N_Y] = \frac{E[e^{-\lambda X_1}]}{1 - E[e^{-\lambda X_1}]}.$$